Non-equilibrium stochastic processes part IIB

1. The velocity of a heavy particle with mass m in a solution fluctuates due to random collisions. The conditional probability $P(v,t) = P(v,t|v_0,0)$ for the particle to have velocity v at time t provided that it has velocity v_0 at time $t=0$ satisfies the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v}\Big[\gamma v + a\frac{\partial}{\partial v}\Big]P\,. \tag{1}$$

Determine the constant a so that the Maxwell-Boltzmann distribution is obtained as an equilibrium distribution. Deduce from the Fokker-Planck equation the time-dependent moments $\langle v(t)\rangle$ and $\langle v(t)^2\rangle$ with initial condition $v(t=0)=v_0$.

2. Consider the branching and annihilation reaction

$$A \xrightarrow{\lambda} A + A \quad \text{and} \quad A + A \xrightarrow{\mu} 0\,. \tag{2}$$

Write down the master equation for the probability $\rho(n,t)$ of observing n molecules at time t. Introduce the variable $x = n/N$ (where N is a large number of molecules) and derive an approximate equation for $P(x,t) = \rho(xN,t)N$. Determine how the number of molecules $\mu(t) = \langle n\rangle$ grows as a function of t.

3. The master equation for the Poisson process reads

$$\frac{\partial \rho_n}{\partial t} = q\big(E^- - 1)\rho_n \tag{3}$$

where $E^- = \exp(-\partial/\partial n)$ and q is the rate at which events occur randomly. Show how to obtain the Poisson distribution by means of a time-dependent WKB approximation. Show also that one obtains a Gaussian approximation to this distribution by expanding the WKB Hamiltonian to second order in momentum p. Discuss how the Gaussian approximates the Poisson distribution, how it fails, and why.

4. Consider a symmetric random walk on a discrete one-dimensional lattice. The corresponding master equation is:

$$\frac{\partial P(j,t)}{\partial t} = [P(j-1,t) + P(j+1,t) - 2P(j,t)]\,. \tag{4}$$

Here $P(j,t)$ denotes the probability that the walker is at site j at time t. The initial condition is $P(j,0) = \delta_{j,0}$. To solve this equation we can use a generating function

$$F(z,t) = \sum_{j=-\infty}^{\infty} z^j P(j,t)\,. \tag{5}$$

Derive the equation for $F(z,t)$ corresponding to the master equation. $F(z,t)$ can be inverted as follows

$$P(j,t) = \frac{1}{2\pi \mathrm{i}} \oint_C \mathrm{d}z z^{-j-1} F(z,t) \tag{6}$$

where C is the unit circle around the origin. Let $z = \exp \mathrm{i}\theta$ and find an expression for $P(z,t)$ in terms of the modified Bessel function defined by

$$I_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \mathrm{e}^{z\cos\theta} \mathrm{e}^{-in\theta} \,. \tag{7}$$

5. Many problems in Biology depend on the arrival of some mobile species of interest at the cell-surface where the species attaches to some receptor. We idealise the cell as a sphere of radius R and consider the case where there is a spherically symmetric distribution $c(r)$ of molecules centred around the cell, with boundary condition $c(r \to \infty) = c_0$. Compute the absorption rate provided that all molecules are adsorbed by receptors on the cell surface.

6. *Path coalescence.* The equation

$$\dot{x} = v \,, \quad \dot{v} = -\gamma v + f(x,t) \tag{8}$$

describes the damped motion of a particle in a fluctuating force $f(x,t)$ (unit mass, $m = 1$). We assume that f is Gaussian with zero mean, correlation time τ, and correlation length ℓ:

$$\langle f \rangle = 0 \,, \quad \langle f(x_1,t_1) f(x_2,t_2) \rangle = f_0^2 \exp[-|t_2 - t_1|/\tau - (x_2 - x_1)^2/2\ell^2] \,. \tag{9}$$

Analyse the path-coalescence transition in the limit $\tau \to 0$ by deriving an equation for the dynamics of small separations Δx and small relative velocities Δv. Derive an equation for the dynamics of $z = \Delta v/\Delta x$ and show that the Lyapunov exponent can be computed as a steady-state expectation value of z. Describe the dynamics of z in qualitative terms.

7. *Brownian motors.* Consider a Brownian motor, performing a random walk along the x-axis. In a simple model the probability that the motor is located on site n at time t satisfies the equation

$$\frac{\partial P(n,t)}{\partial t} = k_r[P(n-1,t) - P(n,t)] + k_l[P(n+1,t) - P(n,t)] \tag{10}$$

where k_r and k_l are the rates for hopping to the left and to the right, respectively. These rates differ due to internal processes such as chemical reactions or conformational changes in the motor. Explain how a Brownian motor can convert fluctuations into directed motion. By multiplying the equation with n and n^2 and summing over n from $-\infty$ to ∞ derive equations of motion for $\langle n$ and $\langle n^2$. Solve these equations and find the time dependence of the mean $\mu(t) = \langle n \rangle$ and the variance $\sigma^2(t) = \langle n^2 \rangle - \langle n \rangle^2$.