Non-equilibrium stochastic processes
Examples sheet 1

1. Path coalescence. This phenomenon is discussed in [1,2]. Consider N independent random walkers, the position of the j-th walker at time t is denoted by $x_j(t)$

$$x_j(t+\delta t) = x_j(t) + f_l\left(x_j(t)\right) \qquad (1)$$

where $t = l\delta t$, $l = 0, 1, 2, 3, \ldots$, and the displacement $f_l(x)$ is taken to be a smooth Gaussian random function with zero mean and correlation length η

$$\langle f_l(x)\rangle = 0\,, \qquad \langle f_l(x) f_{l'}(x')\rangle = \sigma^2\,\delta_{ll'}\,\mathrm{e}^{-(x-x')^2/2\eta^2}\,. \qquad (2)$$

In order to simulate the dynamics on a computer, assume that the N walkers move on the unit interval with periodic boundary conditions. In this case, $f_l(x)$ can be constructed as follows

$$f_l(x) = \sigma\sqrt{\sqrt{2\pi}\,\eta}\sum_k a_{kl}\mathrm{e}^{\mathrm{i}kx-\eta^2k^2/4} \qquad (3)$$

where the sum is over wave vectors of the form $k = 2\pi n$ with $n = -\infty, \ldots, \infty$ and a_{kl} are complex Gaussian random numbers with

$$\langle a_{kl}\rangle = 0\,, \quad \langle a_{kl}a^*_{k'l'}\rangle = \delta_{kk'}\delta_{ll'}\,. \qquad (4)$$

In order for $f_l(x)$ to be real, one must require that $a^*_{kl} = a_{-kl}$. Here the asterisk denotes complex conjugation.

(i) Implement Eq. (3) and show that for given values of l and x, $f_l(x)$ is Gaussian distributed with zero average and variance σ^2.

(ii) Show that in the limit of small η (choose $\eta = 0.1$ for example), the correlation function of $f_l(x)$ is given by Eq. (2). What happens when η is larger?

(iii) Simulate the equation of motion (1) for $N = 100$ particles. What happens? How does the behaviour of the trajectories depend of the parameter values σ and η?

(iv) Numerically determine the Lyapunov exponent λ (lecture notes). Plot λ a function of σ/η. Compare in the same plot your numerical results with the theoretical result (lecture notes). Locate the phase transition (determine the value of σ/η where λ changes sign.

[1] J. Deutsch, J. Phys. A **18**, 1449 (1985)
[2] M. Wilkinson and B. Mehlig, Phys. Rev. E **68**, art. no. 040101 (2003)