Chance and Chaos
Examples sheet 1

1. Path coalescence. This phenomenon is discussed in [1,2]. Consider N independent random walkers, the position of the $j$the walker at time t is denoted by $x_j(t)$

$$x_j(t + \delta t) = x_j(t) + f_l(x_j(t)) \tag{1}$$

where $t = l\delta t$ and $f_l(x)$ is taken to be a Gaussian random function with correlation length ξ

$$\langle f_l(x) \rangle = 0, \qquad \langle f_l(x) f_{l'}(x') \rangle = \sigma^2 \delta_{ll'} e^{-(x-x')^2/2\xi^2}. \tag{2}$$

In order to simulate the dynamics on a computer, assume that the N walkers move on the unit interval with periodic boundary conditions. In this case, $f_l(x)$ can be constructed as follows

$$f_l(x) = \sigma \sqrt{\sqrt{2\pi}\xi} \sum_k a_{kl} e^{ikx - \xi^2 k^2/4} \tag{3}$$

where the sum is over wave vectors of the form $k = 2\pi n$ with $n = -\infty, \ldots, \infty$ and a_{kl} are complex Gaussian random numbers with

$$\langle a_{kl} \rangle = 0, \quad \langle a_{kl} a_{k'l'}^* \rangle = \delta_{kk'} \delta_{ll'}. \tag{4}$$

In order for $f_l(x)$ to be real, one must require that $a_{kl}^* = a_{-kl}$. Here the asterisk denotes complex conjugation. It is convenient to take $a_{0l} = 0$.

(i) Implement eq. (4) and show that for given values of l and x, $f_l(x)$ is Gaussian distributed with zero average and variance σ^2.

(ii) Show that in the limit of small ξ (choose $\xi = 0.1$ for example), the correlation function of $f_l(x)$ is given by eq. (2).

(iii) Simulate the equation of motion (1) for $N = 100$ particles. What happens? How does the behaviour of the trajectories depend of the parameter values σ and ξ? Locate the phase transition.

[1] J. Deutsch, J. Phys. A **18**, 1449 (1985)
[2] M. Wilkinson and B. Mehlig, Phys. Rev. E **68**, art. no. 040101 (2003)